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ON THE SELF-INTERSECTIONS OF FOLIATION CYCLES

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Dedicated to Professor Masahisa Adachi on his 60th birthday

ABSTRACT. The existence of a transverse invariant measure imposes a strong restriction on the transverse complexity of a foliated manifold. The homological self-intersection of the corresponding foliation cycle measures the complexity around its support. In the present paper, the vanishing of the self-intersection is proven under some regularity condition on the measure.

0. INTRODUCTION

It is a basic problem for the theory of foliations to investigate how the transverse structures can be twisted. If a foliation has a foliation cycle, its dynamical behaviour becomes mild in some sense. In this paper, we study the homological self-intersection of foliation cycles which reflects such a phenomenon. For a foliation cycle C, a similar invariant $\chi_\nu(\mathscr{F}) = [C] \cap e(\nu\mathscr{F})$ is considered, where $e(\nu\mathscr{F})$ denotes the euler class of the normal bundle $\nu\mathscr{F}$ to $\mathscr{F}$. The self-intersection $[C]^2$ has some relation to $\chi_\nu(\mathscr{F})$ as explained later. As for the estimate or the vanishing of $\chi_\nu(\mathscr{F})$, see [3 and 5].

The leaves of a foliation $\mathscr{F}$ fill up a foliated manifold $(M, \mathscr{F})$ without intersecting with each other. This suggests the vanishing of the self-intersection in many cases. In this paper a sufficient condition for the vanishing is given.

Throughout this paper, we assume that a foliated manifold $(M, \mathscr{F})$ is both tangentially and transversely oriented, M and $\mathscr{F}$ are smooth, and M is closed. Let p, q, and n be $\dim\mathscr{F}$, $\operatorname{codim}\mathscr{F}$, and $\dim M$ respectively. The tangent (resp. normal) bundle to $\mathscr{F}$ is denoted by $\tau\mathscr{F}$ (resp. $\nu\mathscr{F}$). The foliation cycle (resp. the invariant measure) which corresponds to a transverse invariant measure μ (resp. the foliation cycle C) is denoted by $C = C_\mu$ (resp. $\mu = \mu_C$). Homology and cohomology always imply de Rham's ones. Ω^* denotes the de Rham cochain complex. For the fundamentals of foliation cycles, see [7].

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1. The fundamental classes of compact leaves

First, we deal with the fundamental classes of compact leaves as a special case of foliation cycles. Through some examples, we make simple observations on their self-intersections. To begin with, we review the following two well-known propositions, first of which is known as the Bott vanishing theorem and is one of the starting points of this paper.

Proposition 1.1 [1]. *The normal bundle* $\nu\mathcal{F}$ *admits a* $GL^+(q, \mathbb{R})$*-connection which is flat along the leaves of* $\mathcal{F}$.

Such a connection is called a "*Bott connection*" or a "*basic connection*" and is not unique.

Proposition 1.2 (Milnor-Sullivan-Gromov-Smillie, [4, 6, 2]). *Let* X *be an oriented closed p-manifold and* ξ *be a flat oriented* $\mathbb{R}^p$*-vector bundle over* X. *Then the inequality*

$$|\langle e(\xi), [X]\rangle| \leq 2^{-p}\|X\|$$

holds, where $\|X\|$ *denotes the Gromov invariant of* X. *Especially in the case of* $p = 2$, *the original Milnor inequality has the expression*

$$|\langle e(\xi), [X]\rangle| \leq 2^{-1}\max\{0, -\chi(X)\},$$

where $\chi(X)$ *is the euler characteristic of* X.

Combining these propositions and the next easy one, we obtain Proposition 1.4.

Proposition 1.3. *For a compact leaf* L *of a foliated manifold* $(M, \mathcal{F})$, *the following relation*

$$[L]\cdot[L] = [L] \cap e(\mathcal{F}) \in H_{p-q}(M)$$

holds. If $p = q$, *of course the r.h.s. is nothing but the Kronecker product.*

Proposition 1.4. *Let* L *be a compact leaf of a p-dimensional foliation* $\mathcal{F}$ *on a 2p-dimensional manifold* M. *Then we have*

$$|\langle e(\nu\mathcal{F}), [L]\rangle| = |[L]^2| \leq 2^{-p}\|L\|.$$

Remark. Proposition 1.3 does not hold for general foliation cycles, e.g., if we take $2[L]$ instead of $[L]$, the l.h.s. is multiplied by $2^2 = 4$, but the r.h.s. is multiplied by 2.

Example 1.5. Let us consider a foliated S^2-bundle $\zeta : M^4 \to \Sigma_g$ over a closed oriented surface Σ_g of genus g $(g \geq 2)$. For a given integer J such that $|J| \leq 2^{-1}|\chi(\Sigma_g)| = g - 1$, we can construct a foliated S^2-bundle ζ which has a compact leaf with its self-intersection J as follows. For such J, there exists a flat $SL(2, \mathbb{R})$-vector bundle $\xi : E \to \Sigma_g$ with $\langle e(\xi), [\Sigma_g]\rangle = J$ (see [4]). Let L be the 0-section of ξ. Then, $[L]^2 = J$. Next, consider a flat $\mathbb{R}^3$-bundle $\eta = \xi \oplus \varepsilon^1$ where ε^1 is the trivial flat real line bundle. Then take the associated flat S^2-bundle ζ which is the required one. The subhemisphere bundle ζ_+ (resp. ζ_-) of ζ which corresponds to the hemisphere $\{(x, y, z) \in S^2;\ z > 0$ (resp. $z < 0)\}$ is nothing but the original flat vector bundle ξ (resp. with its orientation reversed) so that the 0-section corresponding to $(0, 0, \pm 1)$ determines a compact leaf $L_\pm$ with $[L_\pm]^2 = \pm J$.

Remark. In the above construction, if we fix the base space Σ_g, the self-intersection of a compact leaf $L_\pm$ is bounded by $g-1$, because the compact leaf is diffeomorphic to the base space. If we want to increase the self-intersection of a compact leaf, we have to embed a compact leaf of higher genus.

One way to do that would be to realize a compact leaf L as a finite orbit of the holonomy group of another compact leaf L_0. However, in such a way, the self-intersection vanishes, because L and L_0 are disjoint and $[L] = k \cdot [L_0] \in H_2(M)$, where k is the covering degree of $L \to L_0$.

Another way would be to realize a compact leaf as a finite orbit of the global holonomy of a foliated bundle. But again, this has little hope as illustrated in the next example.

Example 1.6. Let us consider a foliated S^p-product over a closed oriented p-manifold X, i.e., a product S^p-bundle $\pi : M = X \times S^p \to X$ which has a p-dimensional foliation $\mathscr{F}$ whose leaves are transverse to the fibres. Let L be a compact leaf of $\mathscr{F}$, and k be the degree of the covering $\varphi = \pi|_L : L \to X$. If k is greater than 1, the self-intersection $[L]^2$ vanishes.

Proof. Let $\tilde{\pi} : \widetilde{M} = L \times S^p \to L$ be the pull-back of the foliated S^p-product π by φ so that $\widetilde{M}$ admits a foliation $\widetilde{\mathscr{F}}$ which is transverse to the fibres of $\tilde{\pi}$, i.e., $\widetilde{\mathscr{F}} = \tilde{\varphi}^*\mathscr{F}$ where $\tilde{\varphi} : \widetilde{M} \to M$ is the covering map induced between the total spaces. The inverse image $\tilde{\varphi}^{-1}(L) = \widetilde{L}$ is mapped onto L by $\tilde{\varphi}|_{\widetilde{L}}$ with degree k.

$$\begin{array}{ccc} \widetilde{L} = L_0 \cup L' \subset \widetilde{M} & \xrightarrow{\tilde{\varphi}} & M \supset L \\ \downarrow{\scriptstyle \tilde{\pi}} & & {\scriptstyle \pi}\downarrow \qquad (\varphi \text{ degree } k) \\ L & \xrightarrow{\varphi} & X \end{array}$$

The pull-back $\tilde{\pi}$ has the tautological section whose image L_0 is a connected component of $\widetilde{L}$. Therefore $\widetilde{L}$ splits into a disjoint union of L_0 and $L' = \widetilde{L} \backslash L_0$. They are mapped onto L by $\tilde{\varphi}$ in degree 1 and $(k-1)$ respectively. The pth homology groups and the induced homomorphisms corresponding to the above diagram constitute the following commutative diagram.

$$\begin{array}{ccc} \mathbb{R}\oplus\mathbb{R} & & \mathbb{R}\oplus\mathbb{R} \\ \| & & \| \\ H_p(\widetilde{M}) \simeq H_p(L) \oplus H_p(S^p) & \xrightarrow{(k,1)} & H_p(M) \simeq H_p(X) \oplus H_p(S^p) \\ \downarrow & & \downarrow \\ \mathbb{R} \simeq H_p(L) & \xrightarrow{k} & H_p(X) \simeq \mathbb{R} \end{array}$$

Therefore $[L'] = (k-1)[L_0] \in H_p(\widetilde{M})$ is obtained and we see

$$[L]^2 = [L_0]^2 = (k-1)^{-1}[L'] \cdot [L_0].$$

As seen above, L_0 and L' are disjoint and thus $[L]^2 = 0$ is obtained. □

These observations lead us to the following problem.

Problem. For a given $2p$-dimensional manifold M, does there exist the upper bound for the self-intersection of any compact leaf of any p-dimensional foliation on M? How about in the case of foliated bundles with their base space and fibre specified?

2. Vanishing of self-intersections

2.1. In the previous section we observed an extreme class of foliation cycles and a phenomenon that cycles with larger supports tend to have the trivial self-intersections. In this section, we state and prove a vanishing theorem of the self-intersections for another extreme class of foliation cycles.

Theorem 2.1. *Let μ be a transverse invariant measure of a foliated manifold $(M, \mathscr{F})$ supported on a smooth submanifold N of M satisfying the following conditions.*

(1) $\dim N > p = \dim \mathscr{F}$.

(2) *μ is bounded on N, i.e., for any transverse manifold T of $(N, \mathscr{F}|_N)$, the induced measure $\mu|_T$ on T is bounded w.r.t. any Riemannian volume of T.*

Then the self-intersection of C_μ vanishes, i.e., $[C_\mu]^2 = 0$.

2.2. To prove Theorem 2.1, we construct a closed smooth q-form $\Phi_\mu = \Phi(C_\mu, \varepsilon)$ which is the Poincaré dual to the foliation cycle C_μ and show that $[\Phi_\mu]^2 = 0 \in H^{2q}(M)$.

Let us first consider the case of a compact leaf L and $C_\mu = \int_L$. We fix a Riemannian metric g on M. Hence we consider that the tangent bundle TM of M splits as $TM = \tau\mathscr{F} \oplus \nu\mathscr{F}$ and $\nu\mathscr{F} = \tau\mathscr{F}^\perp$ carries a fibre metric naturally. Let E denote the total space of $\nu\mathscr{F}$ and E_r denote the corresponding disc bundle $\{v \in E\,;\ \|v\| \le r\}$ of radius r. The Thom isomorphism implies that there exists a closed q-form $\Phi \in \Omega^q(E_1, \partial E_1)$ such that $\int_{E_{1,z}} \Phi = 1$ over any point $z \in M$. We fix such a Thom form Φ. Next, we choose $\varepsilon > 0$ so that the exponential map

$$\exp_\varepsilon : E_\varepsilon|_L \to M$$

is an embedding. Let $\mathbf{m}_r$ denote the fibrewise multiplication by r on E, i.e.,

$$\mathbf{m}_r : E \to E, \qquad \mathbf{m}_r(z, v) = (z, r \cdot v).$$

Then, as is well known, the smooth q-form $\Phi(L, \varepsilon) = (\exp_\varepsilon \circ \mathbf{m}_\varepsilon)_* \Phi$ represents the Poincaré dual to $[L]$.

2.3. To obtain the Poincaré dual to foliation cycles C_μ in general, what we want to do is "$\int_{\mathscr{F}}$ [the above construction] $d\mu$." We fix a finite open covering $\mathscr{U} = \{U_i = D_i^p \times D_i^q\,;\ i \in \mathscr{I}\}$ of M by foliated charts and $\{\varphi_i\,;\ i \in \mathscr{I}\}$ be a smooth partition of unity which is subordinate to the covering $\mathscr{U}$. We take $\varepsilon_0 > 0$ so small that $\exp_{\varepsilon_0}$ is an embedding over each plaque of $\mathscr{U}$. Now let Φ_ε denote the q-form $(\mathbf{m}_\varepsilon)_* \Phi$ on E_ε which is also a Thom form and is concentrated on E_ε. Φ_ε is pushed down to M leaf by leaf and is integrated along $\mathscr{F}$ (= the leaf space) by the measure μ. To be precise, for each plaque $D_i^p \times \{y\}$ of U_i $(y \in D_i^q)$, define a smooth q-form $\Phi(y, i, \varepsilon)$ on M as

$$\Phi(y, i, \varepsilon) = \left(\exp_\varepsilon|_{D_i^p \times \{y\}}\right)_* (\varphi_i \circ \pi) \cdot \Phi_\varepsilon$$

where π denotes the bundle projection $E \to M$. Next, for each U_i a smooth q-form $\Phi(i, \varepsilon)$ is defined at each point $z \in M$ as

$$\Phi(i, \varepsilon)(z) = \int_{y \in D_i^q} \Phi(y, i, \varepsilon)(z)\, d\mu_i(y)$$

where μ_i is the measure on D_i^q induced from μ. Finally, we set

$$\Phi(C_\mu, \varepsilon) = \sum_{i \in \mathscr{I}} \Phi(i, \varepsilon).$$

Proposition 2.2. (1) *The q-form* $\Phi(C_\mu, \varepsilon)$ *is closed.*

(2) *The cohomology class of* $\Phi(C_\mu, \varepsilon)$ *does not depend on* ε.

(3) *The closed q-form* $\Phi(C_\mu, \varepsilon)$ *represents the Poincaré dual to the foliation cycle* C_μ.

Proof. (1) We first remark that the form $\Phi(C_\mu, \varepsilon)$ does not depend either on the choice of the covering $\mathscr{U}$ or on that of the partition of unity $\{\varphi_i\}$. Then, on a small neighbourhood of each point, the form $\Phi(C_\mu, \varepsilon)$ is considered to be the integral of closed forms on the neighbourhood along the transverse space by the measure μ. Therefore $\Phi(C_\mu, \varepsilon)$ is closed.

(2) For $0 < \delta < \varepsilon$, Φ_δ and Φ_ε are cohomologuous in E_ε. Hence there exists a $(q-1)$-form $\Psi \in \Omega^{q-1}(E_\varepsilon, \partial E_\varepsilon)$ such that $d\Psi = \Phi_\varepsilon - \Phi_\delta$. Replacing Φ_ε by Ψ in the above construction, we obtain a smooth $(q-1)$-form $\Psi(C_\mu) \in \Omega^{q-1}(M)$ instead of $\Phi(C_\mu, \varepsilon)$. Then it is easy to see that $d\Psi(C_\mu) = \Phi(C_\mu, \varepsilon) - \Phi(C_\mu, \delta)$.

(3) We show that the p-current

$$C(\varepsilon) = \int_M \cdot \wedge \Phi(C_\mu, \varepsilon)$$

(weakly) converges to the current C_μ as $\varepsilon \to 0$. Because the smooth form $\Phi(C_\mu, \varepsilon)$ is closed and its cohomology class is independent of ε, the same holds for $C(\varepsilon)$ and its homology class. Then, especially for any closed p-form $\omega \in \Omega^p(M)$, we have

$$\langle \omega, C(\varepsilon) \rangle = \langle \omega, C_\mu \rangle \quad \text{and} \quad [C(\varepsilon)] = [C_\mu] \in H_p(M).$$

Therefore, it suffices to show the convergence.

By the construction of $\Phi(C_\mu, \varepsilon)$, it is enough to show that

$$\int_M \omega \wedge \Phi(i, \varepsilon) = \int_M \omega \wedge \left(\int_{y \in D_i^q} ((\exp_\varepsilon|_{D_i^p \times \{y\}})_* (\varphi_i \circ \pi) \cdot \Phi_\varepsilon) \, d\mu_i(y) \right)$$

converges to $\langle \varphi_i \cdot \omega, C(\varepsilon) \rangle = \langle \omega, \varphi_i \cdot C_\mu \rangle$ as $\varepsilon \to 0$. By Fubini's theorem, we have

$$\int_M \omega \wedge \Phi(i, \varepsilon) = \int_{y \in D_i^q} \left(\int_M \omega \wedge (\exp_\varepsilon|_{D_i^p \times \{y\}})_* (\varphi_i \circ \pi) \cdot \Phi_\varepsilon \right) d\mu_i(y).$$

Now, we look at the integrand

$$\begin{aligned} F_\varepsilon(y) &= \int_M \omega \wedge (\exp_\varepsilon|_{D_i^p \times \{y\}})_* (\varphi_i \circ \pi) \cdot \Phi_\varepsilon \\ &= \int_{E_1} (\exp_\varepsilon|_{D_i^p \times \{y\}} \circ \mathbf{m}_\varepsilon)^* \omega \wedge (\varphi_i \circ \pi) \cdot \Phi \end{aligned}$$

on D_i^q. In the second expression of $F_\varepsilon(y)$, the norm of $(\exp_\varepsilon|_{D_i^p \times \{y\}} \circ \mathbf{m}_\varepsilon)^*$ is bounded as $\varepsilon \to 0$. Therefore the function $F_\varepsilon(y)$ is bounded on $(0, \varepsilon_0) \times D_i^q$. Moreover, it is clear that the form $(\exp_\varepsilon|_{D_i^p \times \{y\}} \circ \mathbf{m}_\varepsilon)^* \omega$ on E_1 uniformly converges to $(\pi|_{D_i^p\times\{y\}})^*\omega$ as $\varepsilon\to 0$. Then, the limit $F(y)=\lim_{\varepsilon\to 0}F_\varepsilon(y)$ is obtained as

$$\begin{aligned}F(y) &= \int_{E_1}(\pi|_{D_i^p\times\{y\}})^*\omega\wedge(\varphi_i\circ\pi)\cdot\Phi\\ &= \int_{E_1}(\pi|_{D_i^p\times\{y\}})^*(\varphi_i\cdot\omega)\wedge\Phi\\ &= \int_{D_i^p\times\{y\}}\varphi_i\cdot\omega\end{aligned}$$

and turns out to be bounded and continuous on D_i^q. Therefore, by the bounded convergence theorem, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_M\omega\wedge\Phi(i,\varepsilon) &= \lim_{\varepsilon\to 0}\int_{y\in D_i^q}F_\varepsilon(y)\,d\mu_i(y)\\ &= \int_{y\in D_i^q}\lim_{\varepsilon\to 0}F_\varepsilon(y)\,d\mu_i(y)\\ &= \int_{y\in D_i^q}\left(\int_{D_i^p\times\{y\}}\varphi_i\cdot\omega\right)d\mu_i(y)=\langle\varphi_i\cdot\omega\,,\,C_\mu\rangle\end{aligned}$$

and thus the proof is completed. □

2.4. Now, let us prove Theorem 2.1. We may assume $M=N$ because, if not so, we only have to replace $(M,\mathcal{F})$ by $(N,\mathcal{F}|_N)$. What we actually prove is the uniform convergence

$$\lim_{\varepsilon\to 0}\Phi(C_\mu,\varepsilon)\wedge\Phi(C_\mu,\varepsilon)=0.$$

For this, we have to know the behaviour of $\Phi(C_\mu,\varepsilon)$ as $\varepsilon\to 0$ and of the exponential map $\exp_\varepsilon$ a little more closely.

For any point $z\in M$, choose a foliated chart $U=D^p\times D^q=\{(x,y);\ x=(x_1,\dots,x_p),\ y=(y_1,\dots,y_q)\}$ around z as follows. First, let z be the origin $(0,0)$ and L be the leaf through z. Take orthonormal framings $(V_1,\dots,V_p)$ and $(W_1,\dots,W_q)$ of $\tau\mathcal{F}|_L$ and $\nu\mathcal{F}|_L$ respectively around z and take the normal coordinate $(x_1,\dots,x_p)$ of L around z w.r.t. the framing $(V_1(z),\dots,V_p(z))$ and the metric $\mathfrak{g}|_L$. Then, we define the coordinate $(\xi=(\xi_1,\dots,\xi_p),\ \eta=(\eta_1,\dots,\eta_q))$ of $E|_L$ around z as $(\xi,\eta)=(x,\eta_1\cdot W_1(x,0)+\cdots+\eta_q\cdot W_q(x,0))$. Next, take the normal coordinate $(y_1,\dots,y_q)$ of the transverse disk $\exp_{\varepsilon_0}(E_{\varepsilon_0,z})$ w.r.t. the framing $(W_1(z),\dots,W_q(z))$ and the metric $\mathfrak{g}$, and thus as a foliated chart, the y-coordinate is defined around z. At $(0,y)$, the framing $\exp_{*(0,y=\eta)}((V_1,\dots,V_p))=(V_1',\dots,V_p')$ may not be orthonormal. Therefore, by the orthogonal projection, $(V_1',\dots,V_p')$ is projected to the framing $(V_1'',\dots,V_p'')$ of $\tau\mathcal{F}_{(0,y)}$ and, by the Gram-Schmidt method, we obtain an orthonormal framing $(V_1,\dots,V_p)$ of $\tau\mathcal{F}_{(0,y)}$. Then, take the normal coordinate $(x_1,\dots,x_p)$ on the leaf L_y through $(0,y)$ around $(0,y)$ w.r.t. the framing $(V_1,\dots,V_p)_{(0,y)}$ and the metric $\mathfrak{g}|_{L_y}$. Thus we obtain the foliated chart $\{(x,y)\}=D^p\times D^q=U$ of some radii ε_1 and ε_2 of D^p and D^q respectively. The smoothness of $(M,\mathcal{F})$ and the compactness of M guarantees that we can choose ε_1 and ε_2 uniformly, i.e., they do not depend on z but only on $(M,\mathcal{F})$ and $\mathfrak{g}$. We also obtain an orthonormal framing $(V_1, \dots, V_p, W_1, \dots, W_q)$ of $\tau\mathcal{F} \oplus \nu\mathcal{F}|_U = TM|_U$ from the framing $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_q})$ by the Gram-Schmidt method.

Then, by the coordinates (ξ, η) and (x, y), the exponential map

$$\mathrm{E} = \exp_{\varepsilon_2} : E_{\varepsilon_2}|_{D^p \times \{0\}} \to U = D^p \times D^q$$

is expressed as $x_i = \xi_i + f_i(\xi, \eta)$ and

$$y_i = \sum_{l=1}^{q} (\delta_{jl} + a_{jl}(\xi)) \cdot \eta_l + g_j(\xi, \eta)$$

where δ_{jl} denotes Kronecker's symbol, with the estimates

$$f_i(\xi, \eta) \in O(\|\xi\|) \cdot O(\|\eta\|), \quad g_j(\xi, \eta) \in O(\|\xi\|) \cdot O(\|\eta\|^2),$$

and $a_{jl}(\xi) \in O(\|\xi\|)$.

Therefore its differential $D\mathrm{E}$ is expressed around z as

$$D\mathrm{E} = \begin{pmatrix} \delta_{ik} + f_{ik}(\xi, \eta), & f_{il}(\xi, \eta) \\ \sum_l a_{jlk}(\xi) \cdot \eta_l + g_{jk}(\xi, \eta), & \delta_{jl} + a_{jl}(\xi) + g_{jl}(\xi, \eta) \end{pmatrix}$$

with the estimates

$$f_{ik}(\xi, \eta) \in O(\|\eta\|), \quad f_{il}(\xi, \eta) \in O(\|\xi\|),$$
$$a_{j\rho k}(\xi) \in O(1), \quad g_{jk}(\xi, \eta) \in O(\|\eta\|^2),$$

and

$$g_{jl}(\xi, \eta) \in O(\|\xi\|) \cdot O(\|\eta\|).$$

Also, we remark that the framing $(V_1, \dots, V_p)$ (resp. $(W_1, \dots, W_q)$) coincides with $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p})$ (resp. $(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_q})$) at $z = (0, 0)$ and both of them span the same tangent plane $\tau\mathcal{F}$ (resp. $\nu\mathcal{F}$) along $D^p \times \{0\}$, i.e., if we express the second by the linear combination of the first as

$$\frac{\partial}{\partial x_i} = \sum_k (\delta_{ik} + b_{ik}(x, y)) \cdot V_k + \sum_l b'_{il}(x, y) \cdot W_l$$

and

$$\frac{\partial}{\partial y_j} = \sum_k b_{jk}(x, y) \cdot V_k + \sum_l (\delta_{jl} + b'_{il}(x, y)) \cdot W_l,$$

we have the estimates that the functions b_{ik} and b'_{jl} belong to $O(\|(x, y)\|)$ and b'_{il} and b_{jk} belong to $O(\|y\|)$. Of course these functions depend on the choice of the center $z = (0, 0)$ and the orthonormal framing there. But the smoothness and the compactness imply that there exists a constant $K_1 > 0$ which depends only on $(M, \mathcal{F})$ and $\mathfrak{g}$ such that

$$|f_{ik}|, |a_{jlk}|, |b'_{il}|, |b_{jk}| \le K_1 \cdot \|\eta\|, \quad |f_{il}|, |a_{jl}| \le K_1 \cdot \|\xi\|,$$
$$|g_{jk}| \le K_1 \cdot \|\eta\|^2, \quad |g_{jl}| \le K_1 \cdot \|\xi\| \cdot \|\eta\|,$$

and

$$|b_{ik}|, |b'_{jl}| \le K_1 \cdot \|(x, y)\|$$

and η on the r.h.s. can be replaced by y through the map E.

Next, we estimate the form $\Phi_\varepsilon|_{D^p\times\{0\}}$ as $\varepsilon\to 0$. On $E_{\varepsilon_2}|_{D^p\times\{0\}}$, using the coordinate (ξ, η), the q-form $\Phi|_{D^p\times\{0\}}$ is expressed as

$$\Phi_1 = \Phi|_{D^p\times\{0\}} = \sum_{s=0}^{q} \sum_{\substack{|I|=s \\ |J|=q-s}} \alpha_{IJ}(\xi, \eta)\, d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_s}\wedge d\eta_{j_i}\wedge\cdots\wedge d\eta_{j_{q-s}}.$$

Here, $I=(i_1, \ldots, i_s)$ and $J=(j_1, \ldots, j_{q-s})$ are multi-indices such that

$$1\le i_1<\cdots<i_s\le p \quad\text{and}\quad 1\le j_1<\cdots<j_{q-s}\le q$$

and $|I|$ and $|J|$ denote their length. Here again, there exists a constant K_2 which depends only on $(M, \mathscr{F})$ and $\mathfrak{g}$ such that

$$|\alpha_{IJ}(\xi, \eta)|\le K_2.$$

As the multiplication map $\mathbf{m}_\varepsilon$ is simply expressed as $\mathbf{m}_\varepsilon(\xi, \eta)=(\xi, \varepsilon\eta)$, we obtain

$$\Phi_\varepsilon(\xi, \eta)=\sum_s\sum_{I,J}\varepsilon^{-|J|}\alpha_{IJ}(\xi, \varepsilon^{-1}\eta)\, d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_s}\wedge d\eta_{j_1}\wedge\cdots\wedge d\eta_{j_{q-s}}.$$

As our estimate is uniform w.r.t. z, it is enough to estimate the form $\mathrm{E}_*\Phi_\varepsilon$ only on the transverse disk $\{x=0\}=\mathrm{E}(\{\xi=0\})$. Therefore we may assume $f_{il}=0$. Combining all above estimates and this remark, we obtain the following estimate of the form $\mathrm{E}_*\Phi_\varepsilon$.

$$\mathrm{E}_*\Phi_\varepsilon=\sum_s\sum_{I,J}\alpha_{IJ}\varepsilon^{-|J|}\bigwedge_{i\in I}\left(dx_i+\sum_{k=1}^{p}h_{Iik}\,dx_k+\sum_{l=1}^{q}c_{Iil}\,dy_l\right)\wedge\bigwedge_{j\in J}\left(\sum_{l=1}^{q}t_{Jjl}\,dy_l\right).$$

Here, h_{Iik}, c_{Iil}, and t_{Jjl} are some smooth functions such that $h_{Iik}\in O(\|y\|)$, and $c_{Iil}, t_{Jjl}\in O(1)$. Putting the above in order by dx_i's and dy_j's, we obtain

$$\mathrm{E}_*\Phi_\varepsilon=\sum_s\sum_{I,J}\psi_{IJ}\,dx_{i_1}\wedge\cdots\wedge dx_{i_s}\wedge dy_{j_1}\wedge\cdots\wedge dy_{j_{q-s}}$$

where ψ_{IJ} is some function belonging to $O(\varepsilon^{-|J|})$, i.e., there exists some constant K_3 which depends only on $(M, \mathscr{F})$ and $\mathfrak{g}$ such that $|\psi_{IJ}(0, y)|\le K_3\cdot\varepsilon^{-|J|}$. If we use the dual basis of 1-forms $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ to the framing $(V_1, \ldots, V_p, W_1, \ldots, W_q)$, we obtain the similar estimate

$$\mathrm{E}_*\Phi_\varepsilon=\sum_s\sum_{I,J}\varphi_{IJ}\alpha_{i_1}\wedge\cdots\wedge\alpha_{i_s}\wedge\beta_{j_1}\wedge\cdots\wedge\beta_{j_{q-s}}$$

where φ_{IJ}'s satisfy the same condition as ψ_{IJ}'s. Here we introduce the notation

$$\alpha_I=\alpha_{i_1}\wedge\cdots\wedge\alpha_{i_s}\quad\text{and}\quad\beta_J=\beta_{j_1}\wedge\cdots\wedge\beta_{j_{q-s}}.$$

The final estimate is given as follows. To obtain the estimate at a point $m\in M$, take a foliated chart $U=D^p\times D^q=\{(x, y)\}$ around m. For a point $z=(x, y)$ we obtain the above q-form $\mathrm{E}_*\Phi_\varepsilon$ which is now denoted by $\Phi(y, \varepsilon)$. We may assume that there exists a constant $K_4>0$ which depends only on $(M, \mathscr{F})$ and $\mathfrak{g}$ such that if $\|y\|>K_4\cdot\varepsilon$ the point $m=(0, 0)$ is not in the support of $\Phi(y, \varepsilon)$. Thus the Poincaré dual $\Phi(C_\mu, \varepsilon)$ to C_μ around m is given by

$$\Phi(C_\mu, \varepsilon) = \int_{y \in D(K_4 \cdot \varepsilon)} \Phi(y, \varepsilon)\, d\mu(y)$$

where $D(r)$ denotes the subset $\{y;\ \|y\| \leq r\}$ of D^q and $d\mu$ is the measure on D^q determined by μ. Therefore around m, we obtain

$$\begin{aligned}
&\Phi(C_\mu, \varepsilon) \wedge \Phi(C_\mu, \varepsilon) \\
&= \iint_{(y_1, y_2) \in D(K_4 \cdot \varepsilon)^2} \Phi(y_1, \varepsilon) \wedge \Phi(y_2, \varepsilon)\, d\mu \times d\mu(y_1, y_2) \\
&= \iint_{(y_1, y_2) \in D(K_4 \cdot \varepsilon)^2} \left(\sum_{s_1} \sum_{I_1, J_1} \varphi_{I_1 J_1} \alpha_{I_1} \wedge \beta_{J_1} \right) \\
&\qquad \wedge \left(\sum_{s_2} \sum_{I_2, J_2} \varphi_{I_2 J_2} \alpha_{I_2} \wedge \beta_{J_2} \right) d\mu \times d\mu(y_1, y_2) \\
&= \iint_{(y_1, y_2) \in D(K_4 \cdot \varepsilon)^2} \sum_{s_1, s_2} \sum_{I_1, J_1, I_2, J_2} \varphi_{I_1 J_1} \\
&\qquad \cdot \varphi_{I_2 J} \cdot 2\alpha_{I_1} \wedge \beta_{J_1} \wedge \alpha_{I_2} \wedge \beta_{J_2}\, d\mu \times d\mu(y_1, y_2).
\end{aligned}$$

If $|J_1| + |J_2| > q$, $\beta_{J_1} \wedge \beta_{J_2} = 0$. Therefore, putting the above in order, we obtain the following estimate

$$\begin{aligned}
&\Phi(C_\mu, \varepsilon) \wedge \Phi(C_\mu, \varepsilon) \\
&\quad = \iint_{(y_1, y_2) \in D(K_4 \cdot \varepsilon)^2} \sum_{s} \sum_{I, J} \Phi_{IJ} \cdot \alpha_I \wedge \beta_J\, d\mu \times d\mu(y_1, y_2).
\end{aligned}$$

Here, we may assume that there exists a constant K_5 which depends only on $(M, \mathscr{F})$ and $\mathfrak{g}$ such that the functions Φ_{IJ} are estimated as

$$|\Phi_{IJ}| \leq K_5 \cdot \varepsilon^{-|J|}.$$

Therefore, there exists a constant K_6 of the same kind such that the integrand of the above formula is estimated as

$$\left\| \sum_{s} \sum_{I, J} \Phi_{IJ} \cdot \alpha_I \wedge \beta_J \right\| \leq K_6 \cdot \varepsilon^{-q}$$

where the l.h.s. is the natural norm as a q-form. On the other hand, the assumption that μ is bounded against the Lebesgue measure implies that there exists a constant K_7 of the same kind such that $d\mu \times d\mu(D(r) \times D(r))$ is estimated as

$$d\mu \times d\mu(D(r) \times D(r)) \leq K_7 \cdot r^{2q}.$$

Combining the last two estimates, we obtain the uniform estimate

$$\|\Phi(C_\mu, \varepsilon) \wedge \Phi(C_\mu, \varepsilon)\| \leq K_4^{2q} \cdot K_6 \cdot K_7 \cdot \varepsilon^q$$

and thus the uniform convergence

$$\lim_{\varepsilon \to 0} \|\Phi(C_\mu, \varepsilon) \wedge \Phi(C_\mu, \varepsilon)\| = 0$$

is obtained. This completes the proof of Theorem 2.1.

Example 2.3. Let $(M, \mathscr{F}, \mathfrak{h})$ be a Riemannian foliation with a holonomy invariant transverse Riemannian metric $\mathfrak{h}$. Then $\mathfrak{h}$ defines a smooth transverse invariant volume μ and $[C_\mu]^2$ vanishes.

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